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THEORY OF THE UNSYMMETRIC NUTATION DAMPER

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Many different types and configurations of nutation dampers are known, and a single theory capable of describing the action of all of them would be very complicated indeed. However, many of them can be described by considering the effect of a small off-axis weight, mounted so it can rotate around the axis of symmetry of the body on a bearing with friction. The mercury damper can be described in this way when it is so designed that the mercury does not spread out in the channel or break up into beads.

In the derivations which follow, the following postulates are assumed.

(1) The body carrying the damper has an axis of symmetry, and is called a top in this paper.

(2) The damper consists of a small point mass located a distance r from the axis of the top and displaced a distance ℓ along the axis from the c.g. of the top.

(3) The damper can rotate around the axis of the top on a bearing which has small viscous friction.

(4) No external moments are applied to the body or damper mass.

Coordinate System

In order that the time-varying moment of inertia terms be eliminated, the coordinate system chosen rotates with the damper about the axis of the top. It is illustrated below:

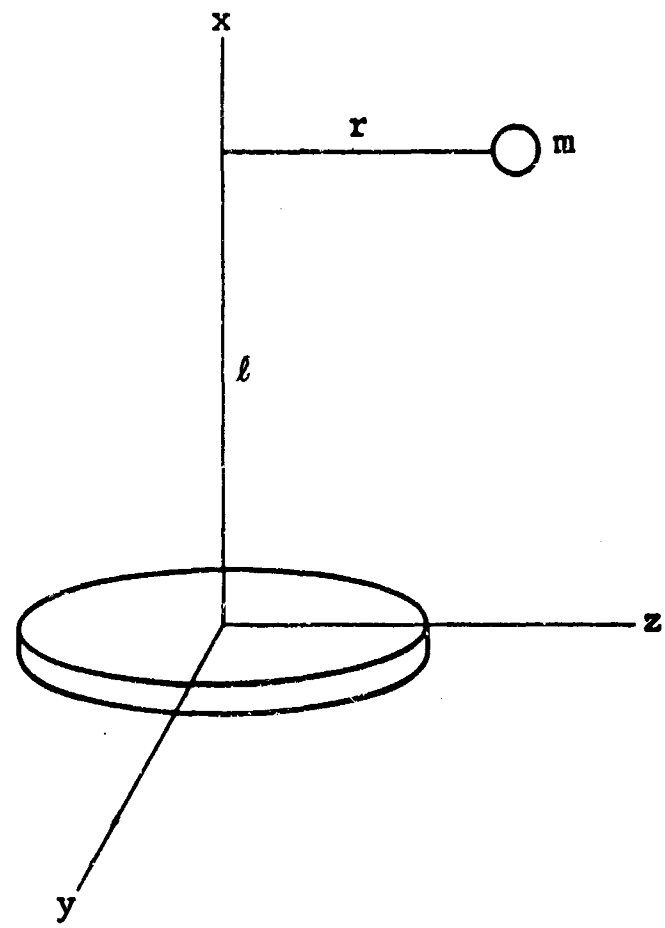

Fig. 1. Moving Coordinate System

The origin of the coordinate system is the c.g. of the top, which has mass M. The angular velocity components of the top are ω_x, ω_y, and ω_z, while the angular velocity components of the coordinate system and damper are Ω, ω_y, and ω_z. The coordinates of the mass are ℓ, 0, and r, while the coordinates of the c.g. of the system are $\frac{m}{M+m}\ell$, 0, $\frac{m}{M+m}r$.

The Lagrangian

Since there are no external moments and the bearing friction is not conservative, the Lagrangian of the system is simply the rotational kinetic energy.

$$T = T_o + \frac{1}{2} M v_M^2 + \frac{1}{2} m v_m^2, \text{ where:}$$

T = total rotational kinetic energy

$T_o = \frac{1}{2} A\omega_x^2 + \frac{1}{2} B\omega_y^2 + \frac{1}{2} B\omega_z^2$ = kinetic energy of top

v_M = velocity of origin of coordinate system

v_m = velocity of damper mass.

<u>Define</u>

$\vec{\Omega} = \Omega\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}$ = angular velocity of coordinate system.

$\vec{r}_m = \ell\vec{i} + r\vec{k}$ = position of damper.

$\vec{r}_{cg} = \frac{m}{M+m}\vec{r}_m$ = position of c.g. of system.

$$v_M = -\vec{\Omega} \times \vec{r}_{cg} = \frac{-m}{M+m}(\vec{\Omega} \times \vec{r}_m)$$

$$v_m = \vec{\Omega} \times (\vec{r}_m - r_{cg}) = \frac{M}{m+M}(\vec{\Omega} \times \vec{r}_m)$$

$$T = \frac{1}{2} A\omega_x^2 + \frac{1}{2} B(\omega_y^2 + \omega_z^2) + \frac{1}{2} M\left(\frac{m}{M+m}\right)^2 (\vec{\Omega} \times \vec{r}_m)^2$$

$$+ \frac{1}{2} m\left(\frac{M}{m+M}\right)^2 (\vec{\Omega} \times \vec{r}_m)^2$$

$$= \frac{1}{2} A\omega_x^2 + \frac{1}{2} B(\omega_y^2 + \omega_z^2) + \frac{1}{2}\bar{m}(\vec{\Omega} \times \vec{r}_m)^2 ,$$

where $\overline{m} = \frac{mM}{m+M}$ = "reduced" mass of damper.

$$\vec{\Omega} \times \vec{r}_m = \vec{i} r\omega_y + \vec{j}\,(\ell\omega_z - r\Omega) + \vec{k}(-\ell\omega_y)$$

$$(\vec{\Omega} \times \vec{r}_m)^2 = r^2\omega_y^2 + \ell^2(\omega_y^2 + \omega_z^2) + r^2\Omega^2 - 2\ell r\omega_z\Omega$$

$$T = \frac{1}{2} A\omega_x^2 + \frac{1}{2}\overline{m}r^2\Omega^2 + \frac{1}{2}(B + \overline{m}\ell^2)(\omega_y^2 + \omega_z^2)$$

$$+ \frac{1}{2}\overline{m}r^2\omega_y^2 - \overline{m}\ell r\omega_z\Omega$$

Define

$$\overline{B} = B + \overline{m}\ell^2$$

$$C = \overline{m}r^2$$

$$D = \overline{m}r\ell$$

$$T = \frac{1}{2} A\omega_x^2 + \frac{1}{2} C\Omega^2 + \frac{1}{2}\overline{B}(\omega_y^2 + \omega_z^2) + \frac{1}{2} C\omega_y^2 - D\omega_z\Omega$$

Euler Angles

The moving coordinate system can be obtained from an inertial coordinate system $x_o y_o z_o$ by three rotations:

(1) An angle ψ about x_o

(2) An angle θ about the new y

(3) An angle φ about the new x

The additional spin of the top can be expressed in terms of an additional angle η about x. The angles are illustrated below:

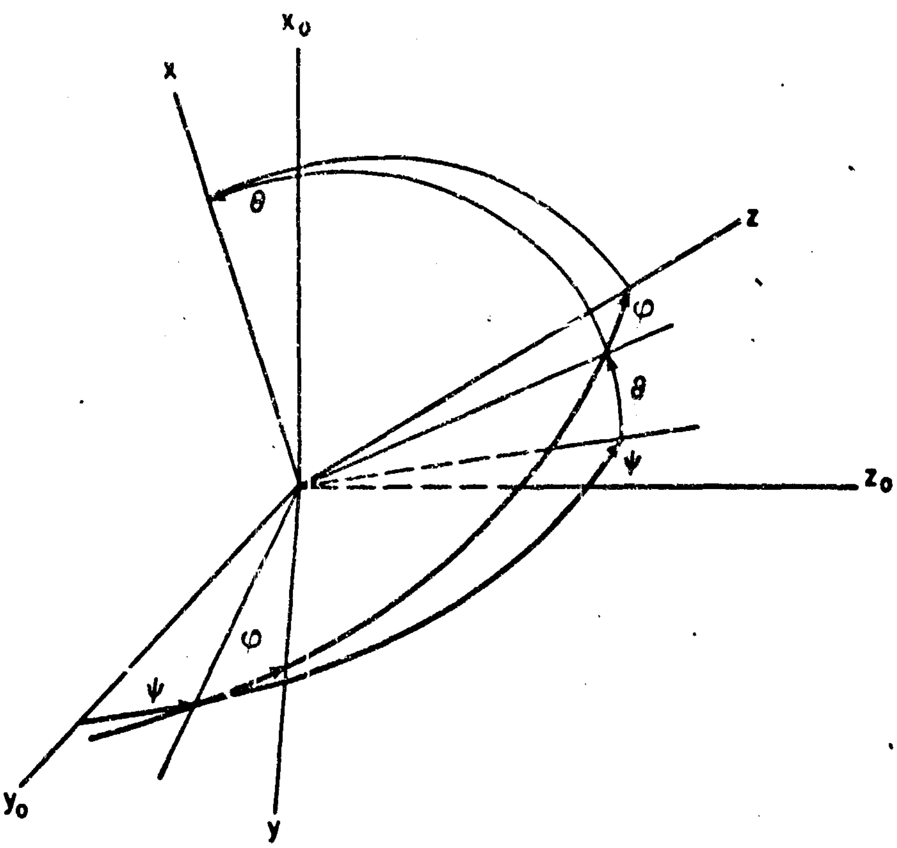

Fig. 2. The Euler Angles

The angular velocities are given in terms of the Euler Angles and η by:

$$\Omega = \dot{\varphi} + \dot{\psi} \cos \theta$$

$$\omega_x = \dot{\eta} + \Omega = \dot{\eta} + \dot{\varphi} + \dot{\psi} \cos \theta$$

$$\omega_y = \dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi$$

$$\omega_z = \dot{\psi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi \quad .$$

The equations of motion are obtained from the kinetic energy by

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = M_{q_i}$$

where the q's are the Euler angles ψ, θ, and φ and the additional angle η. The M_q's are evaluated by the principle of virtual work.

In the case where there are no external moments, all the M_q's vanish except M_η, which, for viscous friction, can be written

$M_\eta = -k\dot{\eta}$, where k is the friction torque coefficient.

The differentiations required to evaluate the equations of motion are somewhat complicated, requiring evaluation of all the partial derivatives of Ω, ω_x, ω_y, ω_z.

The results of this process are:

(1) The η equation

$$A\dot{\omega}_x = -k(\omega_x - \Omega)$$

(2) The φ equation

$$A\dot{\omega}_x + C(\dot{\Omega} - \omega_y \omega_z) - D(\dot{\omega}_z + \Omega\, \omega_y) = 0$$

(3) The θ equation

$$\left[A\omega_x\omega_z + \overline{B}(\dot{\omega}_y - \Omega\,\omega_z) + C(\dot{\omega}_y + \Omega\,\omega_z) + D(\Omega^2 - \omega_z^2)\right]\cos\varphi - \left[-A\omega_x\omega_y + \overline{B}(\dot{\omega}_z + \Omega\,\omega_y) - D(\dot{\Omega} - \omega_y\omega_z)\right]\sin\varphi = 0$$

(4) The ψ equation

$$\cos\theta\left[A\dot{\omega}_x + C(\dot{\Omega} - \omega_y\omega_z) - D(\dot{\omega}_z + \Omega\,\omega_y)\right] + \sin\theta\left[\left\{-A\omega_x\omega_y + \overline{B}(\dot{\omega}_z + \Omega\,\omega_y) - D(\dot{\Omega} - \omega_y\omega_z)\right\}\cos\varphi + \left\{A\omega_x\omega_z + \overline{B}(\dot{\omega}_y - \Omega\,\omega_z) + C(\dot{\omega}_y + \Omega\,\omega_z) + D(\Omega^2 - \omega_z^2)\right\}\sin\varphi\right] = 0$$

Fortunately, it is possible to simplify these equations considerably by combining them so as to eliminate the trigonometric functions of φ and θ.

Combining (1) and (2),

(5) $A\dot{\omega}_x = -k(\omega_x - \Omega)$

(6) $C(\dot{\Omega} - \omega_y\omega_z) - D(\dot{\omega}_z + \Omega\,\omega_y) = k(\omega_z - \Omega)$

Combining (2) and (4),

$$\left[-A\omega_x\omega_y + \overline{B}(\dot{\omega}_z + \Omega\,\omega_y) - D(\dot{\Omega} - \omega_y\omega_z)\right]\cos\varphi$$

$$+\left[A\omega_x\omega_z + \overline{B}(\dot{\omega}_y - \Omega\,\omega_z) + C(\dot{\omega}_y + \Omega\,\omega_z)\right.$$

$$\left.+ D(\Omega^2 - \omega_z^2)\right]\sin\varphi = o$$

Combining this with (3),

$$(7)\quad A\omega_x\omega_z + \overline{B}(\dot{\omega}_y - \Omega\,\omega_z) + C(\dot{\omega}_y + \Omega\,\omega_z) + D(\Omega^2 - \omega_z^2) = o$$

$$(8)\quad -A\omega_x\omega_y + \overline{B}(\dot{\omega}_z + \Omega\,\omega_y) - D(\dot{\Omega} - \omega_y\omega_z) = o$$

Solving these for the highest derivatives,

$$(9)\quad \dot{\omega}_x = \frac{-k}{A}(\omega_x - \Omega)$$

$$\dot{\Omega} - \omega_y\omega_z = \frac{D}{C}(\dot{\omega}_z + \Omega\,\omega_y) + \frac{k}{C}(\omega_x - \Omega)$$

$$\dot{\omega}_z + \Omega\,\omega_y = \frac{D}{B}(\dot{\Omega} - \omega_y\omega_z) + \frac{A}{B}\,\omega_x\omega_y\ .$$

$$(10)\quad \dot{\Omega} = \frac{\overline{B}k}{\overline{B}C - D^2}(\omega_x - \Omega) + \frac{AD}{\overline{B}C - D^2}\,\omega_x\omega_y + \omega_y\omega_z$$

$$(11)\quad \dot{\omega}_z = -\Omega\,\omega_y + \frac{AC}{\overline{B}C - D^2}\,\omega_x\omega_y + \frac{Dk}{\overline{B}C - D^2}(\omega_x - \Omega)$$

$$(12)\quad \dot{\omega}_y = \frac{\overline{B} - C}{\overline{B} + C}\,\Omega\,\omega_z - \frac{A}{\overline{B} + C}\,\omega_x\omega_z - \frac{D}{\overline{B} + C}(\Omega^2 - \omega_z^2)$$

For a small damper and small nutations,

$\omega_y,\ \omega_z << \omega_x,\ \Omega$

$C, D << \overline{E},\ A,\ \overline{B} \cong \overline{B}$

Under these conditions, the above equations reduce to:

(13) $\dot{\omega}_x = 0$

(14) $\dot{\Omega} = -\frac{k}{C}(\Omega - \omega_x) + \frac{AD}{BC}\,\omega_x\omega_y$

(15) $\dot{\omega}_y = \Omega\,\omega_z - \frac{A}{B}\,\omega_x\omega_z - \frac{D}{B}\,\Omega^2$

(16) $\dot{\omega}_z = -\Omega\,\omega_y + \frac{A}{B}\,\omega_x\omega_y$

The effect of these approximations has been shown to be negligible for reasonable dampers by analogue computer simulations.

Transformation to Displacements

Equations (13) to (16) are sufficient to define the angular velocities but it is desirable to determine the angular displacements.

Set $X = \sin\theta\cos\psi$

$Y = \sin\theta\sin\psi$

These coordinates are rectangular distances between the inertial x_o axis and a point unit distance above the c.g. of

the top, on the axis of the top. If, initially, $\psi = \varphi = \theta = o$, the damper is on a line parallel to the -X axis.

Differentiating these expressions,

$$\dot{X} = \dot{\theta} \cos \theta \cos \psi - \dot{\psi} \sin \theta \sin \psi$$

$$\dot{Y} = \dot{\theta} \cos \theta \sin \psi + \dot{\psi} \sin \theta \cos \psi$$

$$\dot{X} \cos \psi + \dot{Y} \sin \psi = \dot{\theta} \cos \theta \cong \dot{\theta} \text{ for small } \theta$$

$$-\dot{X} \sin \psi + \dot{Y} \cos \psi = \dot{\psi} \sin \theta$$

Referring to the Euler angle transformation, the following relations are evident:

$$\dot{\theta} = \omega_y \cos \varphi - \omega_z \sin \varphi$$

$$\dot{\psi} \sin \theta = \omega_y \sin \varphi + \omega_z \cos \varphi$$

$$\Omega = \dot{\varphi} + \dot{\psi} \cos \theta \cong \dot{\varphi} + \dot{\psi}$$

Combining the expressions for $\dot{\theta}$ and $\dot{\psi} \sin \theta$,

$$\dot{X} \cos \psi + \dot{Y} \sin \psi = \omega_y \cos \varphi - \omega_z \sin \varphi$$

$$-\dot{X} \sin \psi + \dot{Y} \cos \psi = \omega_y \sin \varphi + \omega_z \sin \varphi$$

Solving for $\dot{X}$ and $\dot{Y}$

$$\dot{X} = \omega_y \cos (\varphi + \psi) - \omega_z \sin (\varphi + \psi)$$

$$\dot{Y} = \omega_y \sin (\varphi + \psi) + \omega_z \cos (\varphi + \psi)$$

Define

$$\varphi + \psi = R$$

$$\Omega \cong \dot{R}$$

$$\dot{X} = \omega_y \cos R - \omega_z \sin R$$

$$\dot{Y} = \omega_y \sin R + \omega_z \cos R$$

Similarly, solving for ω_y and ω_z,

$$\omega_y = \dot{X} \cos R + \dot{Y} \sin R$$

$$\omega_z = -\dot{X} \sin R + \dot{Y} \cos R$$

$$\dot{\omega}_y = (\ddot{X} + \dot{R}\dot{Y}) \cos R + (\ddot{Y} - \dot{R}\dot{X}) \sin R$$

$$\dot{\omega}_z = -(\ddot{X} + \dot{R}\dot{Y}) \sin R + (\ddot{Y} + \dot{R}\dot{X}) \cos R$$

Substituting in equations (13) - (16) yields, after come simplification:

$$\ddot{R} = -\frac{k}{C}(\dot{R} - \omega_x) + \frac{AD}{BC}\,\omega_x\,(\dot{X} \cos R + \dot{Y} \sin R)$$

$$\ddot{X} + \frac{A}{B}\,\omega_x\dot{Y} = -\frac{D}{B}\,\dot{R}^2 \cos R$$

$$\ddot{Y} - \frac{A}{B}\,\omega_x\dot{X} = -\frac{D}{B}\,\dot{R}^2 \sin R$$

Set $Z = X + iY$, where $i^2 = -1$

The simplified equations of motion become:

$$(17)\quad \ddot{R} = -\frac{k}{C}(\dot{R} - \omega_x) + \frac{AD}{2BC}\,\omega_x(\dot{Z}e^{-iR} + \dot{\bar{Z}}e^{iR})$$

$$(18)\quad \ddot{Z} - i\frac{A}{B}\,\omega_x\dot{Z} = -\frac{D}{B}\dot{R}^2e^{iR}$$

Solutions of the Equations of Motion

Equations (17) and (18) are very nonlinear, so that a general solution cannot be obtained. It is possible to find particular solutions or approximate solutions which will be useful in describing the effects of the damper.

Three cases appear to be useful:

Case I. The damper does not move relative to the top. The solution is a limit cycle.

Case II. The damper oscillates with small amplitudes about the Case I solution. This is called the "slow" damping mode in this report.

Case III. The damper rotates at the nutation frequency. This is called the "fast" damping mode.

Other solutions obtained on the analogue computer appear to be mostly mixtures of Case II and Case III types, but one other is of considerable interest. This is extremely rapid damping which can occur if the initial excitation is of the proper size and phase relative to the damper. This is called "resonant" damping and has not been demonstrated analytically.

Case I: Steady State Solution

Choose $R = \omega_x t$

$$Z = ae^{i\omega_x t}$$

Substitution into equation (17) and (18) shows that (17) is satisfied if a is real and (18) is satisfied if

$$a = \frac{-D}{A-B}$$

$\therefore$ the solution is

$$R = \omega_x t$$

$$Z = -\frac{D}{A-B}\, e^{i\omega_x t}$$

This form should be expected, since the damper behaves like an ordinary unbalance.

Case II: <u>Oscillating Solution</u>

Choose $R = \omega_x t + \alpha$, where α is small

$Z = Z_o + Z_1$, where Z_o is the solution of the homogeneous equation and Z_1 is a small perturbation on this solution.

$$\left|Z_1\right| << \left|Z_o\right|$$

From equation (18)

$$\ddot{Z}_o - i\frac{A}{B}\dot{Z}_o = o$$

$$Z_o = a_o e^{i\frac{A}{B}\omega_x t}$$

For convenience, a_o is chosen to be real.

From equation (17)

$$\ddot{\alpha} \cong -\frac{k}{C}\dot{\alpha} + \frac{AD}{2BC}\, i\frac{A}{B}\, \omega_x^2 a_o \left(e^{i\left(\frac{A-B}{B}\right)\omega_x t} - e^{-i\left(\frac{A-B}{B}\right)\omega_x t} \right)$$

$$= -\frac{k}{C}\dot{\alpha} - \frac{A^2 D}{B^2 C}\omega_x^2 a_o \sin\left(\frac{A-B}{B}\right)\omega_x t$$

$$\ddot{Z}_1 - i\frac{A}{B}\omega_x \dot{Z}_1 \cong -\frac{D}{B}\,\omega_x^2 e^{i\omega_x t} e^{i\alpha}$$

$$\cong -\frac{D}{B}\,\omega_x^2 e^{i\omega_x t}(1 + i\alpha)$$

Solution of these equations is straightforward but somewhat laborious. The results are:

$$\alpha = a_o \frac{A^2 D}{BC(A-B)}\,\omega_x \left(\frac{1}{\left(\frac{k}{C}\right)^2 + \left(\frac{A-B}{B}\right)^2 \omega_x^2} \right) \left(\frac{k}{C} \cos\left(\frac{A-B}{B}\right)\omega_x t \right.$$

$$\left. + \left(\frac{A-B}{B}\right)\omega_x \sin\left(\frac{A-B}{B}\right)\omega_x t \right)$$

$$Z_1 = \frac{-D}{A-B}\left(1 - \frac{iA^2 D a_o \omega_x e^{-i\left(\frac{A-B}{B}\right)\omega_x t}}{4C(A-B)(2B-A)\left(\frac{k}{C} - i\left(\frac{A-B}{B}\right)\omega_x\right)} \right) e^{i\omega_x t}$$

$$- \frac{AD^2 \omega_x^2 a_o}{2(A-B)BC\left(\frac{k}{C} + i\left(\frac{A-B}{B}\right)\omega_x\right)}\, t e^{i\frac{A}{B}\omega_x t}$$

The first term in Z_1 will be recognized as the steady state value with a small oscillation superimposed. This term will be neglected, since it does not influence the damping.

$$Z = Z_o + Z_1$$

$$= a_o \left(1 - \frac{AD^2 \omega_x^2 t}{2(A-B)BC\left(\frac{k}{C} + i\left(\frac{A-B}{B}\right)\omega_x\right)}\right) e^{i\frac{A}{B}\omega_x t}$$

$$|Z| = a_o \left[1 - \frac{AD^2 \omega_x^2 kt}{(A-B)BC^2 \left(\left(\frac{k}{C}\right)^2 + \left(\frac{A-B}{B}\right)^2 \omega_x^2\right)} + \frac{A^2 D^4 \omega_x^4 t^2}{4(A-B)^2 B^2 C^2 \left(\left(\frac{k}{C}\right)^2 + \left(\frac{A-B}{B}\right)^2 \omega_x^2\right)}\right]^{\frac{1}{2}}$$

$$\cong a_o \left(1 - \frac{AD^2 \omega_x^2 kt}{2(A-B)BC^2 \left(\left(\frac{k}{C}\right)^2 + \left(\frac{A-B}{B}\right)^2 \omega_x^2\right)}\right)$$

$$\cong a_o e^{-\frac{t}{tc}}, \text{ where } t_c = \frac{2(A-B)BC^2 \left(\left(\frac{k}{C}\right)^2 + \left(\frac{A-B}{B}\right)^2 \omega_x^2\right)}{AD^2 \omega_x^2 k}$$

Analog computer solutions of equations (17) and (18) indicate that this estimate of t_c is approximately correct.

Case III: Nutation Frequency Solution

Choose $R = \frac{A}{B}\,\omega_x t + \varphi$, where φ is a constant.

$$Z = Z_o + Z_1, \text{ where } |Z_1| << |Z_o|$$

$$Z_o = a_o e^{i\frac{A}{B}\omega_x t}$$

From equation (17)

$$\ddot{R} = o = \frac{-k}{C}\left(\frac{A-B}{B}\right)\,\omega_x + \frac{AD}{2BC}\, i\frac{A}{B}\,\omega_x^2 a_o\left(e^{+i\varphi} - e^{i\varphi}\right)$$

$$\sin\varphi = \frac{k(A-B)B}{A^2 D a_o \omega_x}$$

Since $\sin\varphi \leq 1$, $a_o \geq \frac{k(A-B)B}{A^2 D\omega_x}$ in order that this solution exist.

From equation (18)

$$\ddot{Z}_1 - i\frac{A}{B}\dot{Z}_1 = -\frac{D}{B}\left(\frac{A}{B}\,\omega_x\right)^2 e^{i\frac{A}{B}\omega_x t} e^{i\varphi} = -\frac{DA^2}{B^3}\,\omega_x^2 e^{i\varphi} e^{i\frac{A}{B}\omega_x t}$$

Since $e^{i\frac{A}{B}\omega_x t}$ is a solution of the homogeneous equation, choose

$$Z_1 = a_1 t e^{i\frac{A}{B}\omega_x t}$$

Substituting this form into the differential equation;

$$a_1 = i\frac{DA}{B^2}\,\omega_x e^{i\varphi} = -\frac{DA}{B^2}\,\omega_x\,(\sin\varphi - i\cos\varphi)$$

$$= \frac{-DA}{B^2}\,\omega_x\left(\frac{k(A-B)B}{A^2 Da_o\omega_x} + i\sqrt{1 - \frac{k^2(A-B)^2B^2}{A^4D^2a_o^2\omega_x^2}}\right)$$

$$Z = Z_o + Z_1$$

$$= \left[a_o - \frac{DA}{B^2}\,\omega_x t\left(\frac{k(A-B)B}{A^2Da_o\omega_x} + i\sqrt{1 - \left(\frac{k(A-B)B}{A^2Da_o\omega_x}\right)^2}\right)\right]e^{i\frac{A}{B}\omega_x t}$$

$$|Z| = \left(a_o^2 - \frac{2k(A-B)t}{AB} + \left(\frac{DA}{B^2}\,\omega_x\right)^2 t^2\right)^{\frac{1}{2}}$$

Since a_o is not a common factor in this expression no assumption of an exponential form for the damping is reasonable. Analog computer studies of the equations and this solution indicate that it is valid for a short time, but that a much better approximation is obtained by "starting over" after each nutation cycle.

Define

$$\Delta t_n = \text{time per nutation cycle}$$

$$= 2\pi \frac{B}{A\omega_x}$$

$$|z|^2_{t=o} = a_o^2$$

$$|z|^2_{t=\Delta t_n} = a_o^2 - \frac{2k(A-B)\Delta t_n}{AB} + \left(\frac{DA}{B^2}\,\omega_x\right)^2 \Delta t_n^2$$

$$|\dot{z}|^2 \cong \frac{|z|^2_{t=\Delta t_n} - |z|^2_{t=o}}{\Delta t_n} = \frac{-2k(A-B)}{AB} + \left(\frac{DA}{B^2}\,\omega_x\right)\Delta t_n$$

$$= \frac{-2k(A-B)}{AB} + \frac{2\pi D^2 A\omega_x}{B^3}$$

$$|z|^2 \cong a_o^2 - \left(\frac{2k(A-B)}{AB} - \frac{2\pi D^2 A\omega_x}{B^3}\right)t$$

The analog computer solutions indicate that this form is reasonably accurate until the amplitude has decayed to quite near the critical value.

Distribution:

406	40605
40601	4063
40602	4064
40603	4065
40604	